General Topology and its Applications 2 (1972) 1–10. North-Holland Publishing Company

ORDERABLE TOPOLOGICAL SPACES

M. VENKATARAMAN,
M. RAJAGOPALAN and T. SOUNDARARAJAN
Department of Mathematics, Madurai University, Madurai–2, Tamil Nadu, India

Received 5 January 1971

AMS Subj. Class.: Primary 54F05, 06A45, 22A99, 54D35

0. Introduction

A topological space (X, T) is said to be orderable if there exists a total order $>$ on X such that the order topology on X given by $>$ coincides with the topology T. In this note, we are primarily interested in (1) when the underlying space of a topological group is orderable (the order need not be compatible with the group operations) and (2) when the Stone-Čech compactification of a topological space is orderable. Some of the results proved are:

(1). Let (G, T) be a topological group which is not totally disconnected. Then the topology T is orderable if and only if (G, T) contains the additive topological group of real numbers R as an open subgroup.

(2). Let (G, T) be a locally compact, totally disconnected, nondiscrete topological group. Then the topology T is orderable if and only if (G, T) contains an open subgroup H which as a topological space is homeomorphic to the Cantor set.

(3). If X is Hausdorff and completely regular and its Stone-Čech compactification βX is orderable, then X must be countably compact.

(4). Let X be a completely regular Hausdorff space which is not compact. If X is metric, Lindelöf, paracompact or separable, then βX is not orderable.

Proofs of some of the results in this paper were omitted at the suggestion of the referee.

Section 1 contains some definitions, known and easy theorems on orderable spaces. Section 2 contains results on topological groups and homogeneous spaces. Section 3 deals with orderability of the Stone-Čech compactification. Section 4 gives a list of spaces which are not orderable. Section 5 concludes with some remarks and some examples.

1. Definitions and theorems

Definition 1.1. If $(X, >)$ is a totally ordered set, the order topology on X given by $>$ is the topology which has a subbase F of open sets defined as follows: A subset A of X is an element of F if and only if there exists an element $a \in X$ such that either $A = \{x \in X \mid a > x\}$ or $A = \{x \in X \mid x > a\}$.

Definition 1.2. A topological space (X, T) is said to be orderable if there exists a total order $>$ on X such that the order topology given by $>$ on X (by Definition 1.1) coincides with the topology T. It is a surprising and well known fact that a subspace of an orderable space need not be orderable.

The following proposition is well known [3]:

Proposition 1.3. Let $(X, >)$ be a totally ordered set and let T be the order topology on X induced by $>$. Then the following hold:

(1). If A is a compact subset of X (that is of (X, T)) then the relative topology on A coincides with the topology on A given by the induced order on A. Also A has a least element and a greatest element.

(2). If B is a connected subset of X with at least two elements then: (a) whenever $x, y \in B$ and $x > z > y$ for some $z \in X$, then $z \in B$; (b) if $x, y \in B$ and $x > y$, there is always a $z \in B$ such that $x > z > y$; (c) the relative topology on B coincides with the topology given by the induced order on B; (d) B must be either an interval or a ray.

(3). X is connected if and only if X has no jumps and is order complete.

(4). X is compact if and only if it has a least and a greatest element and is order complete.

Corollary 1.4. If (X, T) is an orderable space and $A \subset X$ is either compact or connected, then the subspace A is orderable.

Proposition 1.5. Any discrete space X is orderable.
Proof. See [4]. [*Hint*: If $|X| = \aleph_\alpha$, consider the lexicographically ordered set $[1, \omega_\alpha) \times Z$ where Z is the ordered set of all integers (positive and negative) and ω_α is the smallest ordinal of cardinal $\aleph_\alpha$].

Proposition 1.6. Let (X, T) be a compact connected homogeneous space with at least two elements. Then (X, T) is not orderable.

Notation 1.7. Let $(X, >)$ be an ordered set and x, y be two elements of X. Then

$$\begin{aligned}
(x,y) &= \{z \in X \mid y > z > x\} \quad \text{(called *open interval*)},\\
(x,y] &= \{z \in X \mid y \geqslant z > x\},\\
[x,y) &= \{z \in X \mid y > z \geqslant x\},\\
[x,y] &= \{z \in X \mid y \geqslant z \geqslant x\} \quad \text{(called *closed interval*)}.
\end{aligned}$$

Note that (x,y) is an open set and $[x,y]$ is a closed set for the order topology on X given by $>$.

Proposition 1.8. Let (X, T) be an orderable topological space and having a topologically dense subset S with $|S| = \theta$ $(\theta \geqslant \aleph_0)$. Then (a) at each point x of X there is a basis F of neighborhoods of x such that $|F| \leqslant \theta$ and (b) $|X| \leqslant 2^\theta$.

Corollary 1.9. If (X, T) is a separable topological space and orderable then (X, T) satisfies the first axiom of countability and $|X| \leqslant c$.

Proposition 1.10. Let (X, T) be an orderable topological space ordered by $>$. Let $x \in X$ have a family of neighborhoods F such that $|F| = \aleph$ $(\geqslant \aleph_0)$ and $\bigcap_{U \in F} U = \{x\}$. Then there exists a family F_1 of neighborhoods of x such that $|F_1| = \aleph$ and F_1 is a basis of neighborhoods at x.

2. Topological groups

In this section, we consider the characterization of some topological groups which are orderable as topological spaces. We first have:

Lemma 2.1. Let (X, T) and (Y, T') be topological spaces and $f: (X, T) \to (Y, T')$ be an open onto map. Let for each $y \in Y$, $f^{-1}(y)$ contain a connected subset with at least two elements. Then (Y, T') must be discrete if (X, T) is orderable.

Proof. This follows from the fact that for each $y, f^{-1}(y)$ contains a non-empty open interval.

Proposition 2.2. Let (G, T) be a topological group and H be a connected subgroup with at least two distinct elements. If (G, T) is orderable as a topological space then H must be an open subgroup of G.
Proof. Consider the equivalence relation R defined on G by $x\ R\ y$ if $y \in xH$ (= the equivalence class containing x). Consider the set G/R and give it the quotient topology for the map P: $G \to G/R$ defined by $P(x) = xH$. If O is open in G, then $P^{-1}(P(O)) = OH$. Since G is a topological group, OH is an open set and hence $P(O)$ is open in X/R. Thus P is an open map. Already P is continuous and onto. Also since H is connected with at least two distinct elements and G is a topological group, each xH is also connected and has at least two elements. If (G, T) is orderable, then Lemma 2.1 applies and yields that G/R is discrete. Hence if e is the identity of G, $P(e)$ must be open in G/R and hence $P^{-1}(P(e))$ must be open in G. But $P^{-1}(P(e)) = H$. Hence H is open in G.

Proposition 2.3. Let (X, T) be a topological space having a connected subset A with at least two distinct elements. Let Y be any topological space. If the product space $X \times Y$ is orderable, then Y must be discrete.
Proof. This follows from Lemma 2.1 by considering the projection map from $X \times Y$ to Y and using the hypothesis.

Theorem 2.4. Let (G, T) be any topological group which is not totally disconnected. Then (G, T) is orderable as a topological space if and only if (G, T) contains an open (normal) subgroup H topologically isomorphic with R, the additive group of real numbers with usual topology.
Proof. *Necessity.* Suppose (G, T) is orderable as a topological space. Let G_0 be the connected component at the identity e of G. Then G_0 is a connected normal subgroup with at least two elements since G is not totally disconnected. Then by Proposition 2.2, G_0 is an open sungroup of G. We have only to show that G_0 (with relative topology) is topologically isomorphic with R. We shall denote G_0 with relative topology by (G_0, T).

By Proposition 1.3, the topological group (G_0, T) is orderable as a topological space and is connected with at least two elements. It is also homogeneous. Also (G_0, T) is a Hausdorff space, since any orderable space is Hausdorff. G_0 is locally compact [1]. For, given any x, the closed intervals containing x form a basis of neighborhoods of x since G_0 is connected and each of these closed intervals is compact by Proposition 1.3. Hence, (G_0, T) is a connected locally compact Hausdorff topological group. Also, if H is a compact connected subgroup of (G_0, T), then H is a compact connected homogeneous orderable space and hence by Proposition 1.6 must be the identity alone. Thus (G_0, T) has maximal compact connected subgroups equal to the identity. Hence, by a theorem in [8], G_0 must contain a subgroup H, topologically isomorphic with R. Then H is a connected subgroup of G_0 and so by Proposition 2.2, H is open in G_0. An open subgroup is closed also. But G_0 is connected. Hence $H = G_0$.

Sufficiency. (G, T) is a topological group having an open subgroup H topologically isomorphic with R. Then H is orderable, and now the space (G, T) is a topological sum of spaces each homeomorphic with H (namely the cosets of H). Each of these spaces is orderable with no end points. That (G, T) is orderable follows by Theorem 9 of [4].

Theorem 2.5. Let (G, T) be an infinite, locally compact, totally disconnected topological group. Then (G, T) is orderable as a topological space if and only if either G is discrete or G contains an open subgroup H which as a topological space is homeomorphic with the Cantor set with its usual topology.

Proof. *Necessity*. Suppose (G, T) is orderable as a topological space. Then (G, T) is Hausdorff, and since it is locally compact and totally disconnected, it has a basis of neighborhoods at the identity consisting of compact open subgroups [8]. Let H be a compact open subgroup. If H is finite and consists of $e, x_1, ..., x_n$, then it is easily shown, using the Hausdorff property, that $\{e\}$ is open and so G is discrete.

Suppose H is infinite. Then H is a compact totally disconnected topological group. By Proposition 1.3, H is orderable as a topological space. Also H is homeomorphic as a topological space to $\{0,1\}^{\aleph}$, where $\{0,1\}$ is a discrete two point space and $\aleph$ is some cardinal [6]. If $\aleph > \aleph_0$, then $\{0,1\}^{\aleph}$ contains $\{0,1\}^{\aleph_1}$ as a closed and compact subspace. Hence $\{0,1\}^{\aleph_1}$ is orderable. But this space is separable since $\{0,1\}$ it self is separable [2]. Hence $\{0,1\}^{\aleph_1}$ satisfies the first axiom of countability by Proposition 1.9. This is a contradiction. Hence H is homeomorphic to $\{0,1\}^{\aleph_0}$. But this, as is well known, is homeomorphic to the Cantor set with the usual topology. This establishes the necessity.

Sufficiency. If G is discrete, then it is orderable by Proposition 1.5. If G has an open subgroup H topologically homeomorphic with the Cantor set, then H is orderable and G is a topological sum of spaces each homeomorphic with H. Since H is compact, it has end points under any total order giving the topology. That (G, T) is orderable follows now by Theorem 9 of [4].

Theorem 2.6. Let (G, T) be a separable totally disconnected topological group. Then (G, T) is orderable as a topological space if and only if it is metrizable and zero-dimensional.

Proof. Suppose (G, T) is orderable as a topological space. Then by Proposition 1.8, (G, T) satisfies the first axiom of countability. Hence (G, T) is metrizable [8]. That a metric totally disconnected orderable space is zero-dimensional has been proved in [5]. Conversely, that a separable metric zero-dimensional space is orderable has been proved in [7].

Theorem 2.7. Let (G, T) be a totally disconnected topological group such that the identity element is a G_δ. Then (G, T) is orderable as a topological space if and only if (G, T) is metric and strongly zero-dimensional.

Proof. Let (G, T) be orderable as a topological space by $>$, say. Then by Proposition 1.10, the first axiom of countability is satisfied at e. Hence the first axiom of countability is satisfied by G since G is homogeneous. Hence it follows that (G, T) is metrizable. It is also totally disconnected and orderable. That (G, T) is strongly zero-dimensional follows by the theorem of [5].

Conversely, suppose (G, T) is metric and strongly zero-dimensional. Then a theorem of [5] asserts that (G, T) is orderable.

3. Orderability of the Stone-Čech compactification

Let X be a Hausdorff completely regular space. In this section, we are interested in βX, the Stone-Čech compactification of X, being orderable. We start with the well known

Proposition 3.1. If N is the discrete space of positive integers, then βN is not orderable.

Proof. It is known that $|\beta N| = 2^c$ and βN is obviously separable. Hence by Proposition 1.9, βN cannot be orderable.

Proposition 3.2. Let X be a completely regular Hausdorff space. If βX is orderable then X must be normal and pseudo compact and hence must be countably compact.

Proof. Let βX be orderable. It is well known that any orderable space is completely normal. Hence X is normal. Suppose X is not pseudo-compact. Then there exists a continuous real-valued function f on X which is unbounded. By considering $|f|$, if necessary we can assume that f is non-negative. Then we can find a sequence $x_1, x_2, \ldots$ in X such that $f(x_n) - f(x_{n-1}) > 1$ (since f is unbounded). It is easily checked that the subset $A = \{x_1, \ldots, x_n, \ldots\}$ in X is closed and discrete as a subspace. Hence it is homeomorphic to N (the set of positive integers). Let us consider $\mathrm{cl}A$ in βX. Any bounded continuous real valued function on A can by Tietze's theorem be extended to a continuous bounded real-valued function f_1 on X. This f_1 can in turn be extended to a bounded continuous real-valued function f_2 on βX. Then $f_2|\mathrm{cl}A$ is a bounded continuous real valued extension of the function f on A. Hence we get that $\mathrm{cl}A$ in βX is βA. But A is homeomorphic to N. Hence $\mathrm{cl}A$ is homeomorphic to βN. Since βX is orderable, any compact subset of it must be orderable by Proposition 1.3. Hence $\mathrm{cl}A$ and hence βN is orderable. This contradicts Proposition 3.1. Hence X is pseudo-compact. Already X is normal. We know that a normal pseudo-compact space is countably compact [2], [3]. This completes the proof of the proposition.

Proposition 3.3. Let X be a non-compact completely regular Hausdorff space. Then βX is not orderable, if any one of the following conditions is satisfied:

(1). X is metrizable.
(2). X is paracompact.
(3). X is σ-compact.
(4). X is Lindelöf.
(5). X is separable.

Proof. Since X is not compact, $\beta X \backslash X \neq \emptyset$. Suppose βX is orderable. Then by Proposition 3.2, X must be countably compact. But a countably compact paracompact space is compact. Hence X cannot satisfy (2). Since a metric space is paracompact, X cannot satisfy (1) either. Since X is regular and a regular Lindelöf space is paracompact, X cannot satisfy (4). Since (3) implies (4), X cannot satisfy (3) either. If X is separable, βX is also separable and hence, by Proposition 1.9, βX satisfies the first axiom of countability. But at no point of $\beta X \backslash X$ the first axiom of countability is satisfied [3]. This is a contradiction.

Proposition 3.4. Let X be a connected space which is dense in another space Y. Then if Y is orderable, X is orderable and $|Y \setminus X| \leqslant 2$. In particular, if for a connected completely regular Hausdorff space βX is orderable, then $|\beta X \setminus X| \leqslant 2$, X is orderable, and if X is ordered by $>$ then for any continuous real-valued function f on X, there exist two points a and b with $a < b$ such that f is constant on L_a and R_b.

Remark. There are connected orderable spaces X such that βX is orderable. For example, $X = [1,\Omega)\times[0,1]$ with lexicographic ordering and for each ordinal α, the two points $(\alpha, 1)$ and $(\alpha+1,0)$ identified [3].

4. Non-orderable spaces

The following theorem lists at a single place axamples of several non-orderable spaces. Since a finite Hausdorff space is orderable, we deal with infinite spaces only.

Proposition 4.1. The following spaces are not orderable:

(1) $\beta N \setminus N$, βQ, βR, $\beta Q \setminus Q$, $\beta R \setminus R$, where Q is the space of rational numbers and R the space of real numbers;

(2) any βX of a space X such that βX contains a zero set Z and a set E such that Z meets $\mathrm{cl}\, E$ but not $X \cup E$;

(3) any βX which has a zero set disjoint from X;

(4) any βX of a space X such that $\beta X \setminus \upsilon X \neq \emptyset$, where υX is the real compactification of X;

(5) any βX of a locally compact real compact but not compact space X;

(6) any compact extremely disconnected space;

(7) any compact basically disconnected space;

(8) βX of a P-space X;

(9) βX of a locally compact F-space X such that $\beta X \setminus X$ is infinite;

(10) any compact F-space;

(11) any non-discrete locally compact F-space.

Proof. The proofs of the results (1)–(10) follow easily from either the results of section 3 or each of them contains a copy of βN (This is justified by appropriate results from [3]) Regarding (11), if $x \in X$ is not isolated and X is orderable, then x has an infinite compact neighbourhood, and in this we can choose an increasing sequence (or decreasing sequence) of distinct elements, and due to compactness this sequence has to converge. But in an F-space no sequence converges unless it is eventually a constant [3]. This is a contradiction. Hence the proposition follows.

5. Remarks and examples

Remark 5.1. If an orderable space X has a countable topologically dense subset then it need not have a countable order dense subset. For example, the ordered set obtained from the real numbers by splitting each irrational number into two consecutive elements.

Remark 5.2. If $X \times Y$ is orderable and Y is discrete, then X need not be orderable. Let X be the space $(0,1) \cup \{2\}$. This space is not orderable. Let Y be a countably infinite discrete space. Then $X \times Y$ is orderable.

Remark 5.3. An open and closed subset of an orderable space need not be orderable. For in the example of Remark 5.2, any $X \times \{y\}$ is an open and closed subset homeomorphic to X.

Remark 5.4. If $X \times Y$ is orderable and X is compact or connected then X is orderable (by Proposition 1.3).

Remark 5.5. Let Y be a topological space. Then Y is discrete if and only if $Y \times X$ is orderable for every orderable space X. If $Y \times [0,1]$ is orderable, then Y becomes discrete by Proposition 2.3. If Y is discrete and X orderable then $Y \times X$ becomes orderable by Theorem 9 of [4].

Remark 5.6. Let (G,T) be a topological group which is not totally disconnected. Suppose it is orderable. Then by Theorem 2.4, R is an open subgroup of G. As a topological space (G,T) is homeomorphic to $R \times$ (a discrete space). We cannot expect R to be a direct summand of G. For let (G, T) be the topological group of all non-singular real matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $b \in R$ and a is of the form 2^m, $m \in Z$. Here the subset of all matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ is an open subgroup topologically isomorphic with R.

Remark 5.7. In both Theorems 2.4 and 2.5, it is easy to show that if (G, T) is orderable as a topological space, then (G, T) must be metrizable [6]. From Theorem 2.5 we can deduce that the group of all p-adic numbers is orderable as a topological space since the p-adic integers is a compact open subgroup which as a topological space is homeomorphic to $\{0,1\}^{\aleph_0}$.

Remark 5.8. Theorem 2.6 along with Lynn's theorem [7] yields that any subspace of an orderable separable totally disconnected topological group is also orderable. In particular, any subspace of the rationals or the Cantor set is orderable.

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